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RESEARCH ARTICLE

Effect of the source term on steady free convection boundary layer flows over a vertical plate in a porous medium. Part II

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Abstract The problem of steady free convection boundary layer over a vertical isothermal impermeable flat plate which is embedded in a fluid-saturated porous medium with volumetric heat generation or absorption is studied in this paper using the Darcy equation model. The case of the *externally prescribed* source terms $S = S(x, y)$ is considered in this paper. It is shown that the corresponding boundary value problem depends on the sign of the plate temperature, which implies that the source term breaks the usual upflow or downflow symmetry of the free convection problem. Looking for similarity solutions, analytical and numerical solutions of the transformed boundary value problem are obtained for several values of the problem parameters. It is also shown that, contrary to the widely spread opinion, the exponential form of the internal heat generation term is not a necessary requirement of similarity reduction.

Keywords Internal heat generation · Similar flows · Porous media · Darcy free convection · Exact solutions

Nomenclature

c_p specific heat at constant pressure.
f similar stream function.
g magnitude of the acceleration due to gravity.
G dimensionless function, Eq. (12).
I_j modified Bessel function.

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k_m	effective thermal conductivity.
K	permeability of the porous medium.
K_j	modified Bessel function.
L	characteristic length.
r_0	dimensionless quantity, Eq. (11b).
Ra	Darcy-Rayleigh number.
S	heat source or sink.
S_0	intensity scale of the sources or sinks, Eq. (11a).
$\tilde{S}$	dimensionless source function, Eq. (11a).
$s_g,\ s_T$	sign functions.
T	fluid temperature.
T_0	specific temperature scale.
T_w	wall temperature.
T_∞	ambient temperature.
u, v	velocity components along x- and y- axes.
x, y	Cartesian coordinates along the plate and normal to it, respectively.
X, Y	dimensionless Cartesian coordinates.
z	intermediate variable.

Greek symbols

α	effective thermal diffusivity.
β	coefficient of thermal expansion.
η	similarity variable.
λ	power law exponent of the wall temperature variation.
θ	dimensionless temperature.
ρ	fluid density.
υ	kinematic viscosity.
ψ	stream function.

1. Introduction

The present paper is the second part of Magyari et al. (2007), called hereinafter Part I, where the effect of the source term (volumetric heat generation or absorption) in the Darcy free convection boundary layer flow over a vertical flat plate was studied. In Part I it was assumed that the heat generation or absorption takes place in a *self-consistent* way, the source term $q''' \equiv S$ of the energy equation being an analytical function of the local temperature difference $T - T_\infty$. It was shown that due to the presence of S, the physical equivalence of the up- and downflows gets in general broken, in the sense that the free convection flow over the upward projecting hot plate ("upflow") and over its downward projecting cold counterpart ("downflow") in general become physically distinct. Considering different forms of S, the consequences of this circumstance were examined and several analytical solutions were derived. Some of them describe *algebraically decaying* boundary layers, which can also be recovered as limiting cases of *exponentially decaying* ones.

Consulting the literature devoted to this field, we remark that internal heat generation was taken into account by introducing an exponential decaying heat generation term in similarity like formulations. For free convection flow over a vertical surface immersed in a viscous fluid, Crepeau and Clarksean (1997) were able to find similarity solutions when internal heat generation sources are present in the flow domain. Then, in a series of papers, Postelnicu and Pop (1999), Postelnicu et al. (2000), Grosan and Pop (2001, 2002), Bagai (2003) studied the free convection over a vertical or horizontal flat plate or over a body of arbitrary shape embedded in a fluid-saturated porous medium by using the Darcy or non-Darcy flow models. In direct connection with the present paper is the contribution by Postelnicu and Pop (1999), where numerical solutions of the problem of free convection boundary layer over a vertical surface embedded in a fluid-saturated porous medium have been presented. The wall temperature distribution was taken there as proportional to x^{λ}, where x is the distance measured along the plate and λ is a constant, and an exponential decaying heat generation term was considered.

The main aim of this paper is to analyze the free convection over an isothermal and impermeable vertical flat plate which is embedded in a fluid-saturated porous medium with internal heat generation or absorption sources $S[W/m^3]$, depending on the spatial coordinates x, and *y explicitly*. These sources, called in Part I, sources of type (II), are in general *externally controlled*, their space distribution and intensity being *prescribed*, or they result from some physical laws which do not depend on temperature directly, as e.g., the heat generation by absorption of infrared radiation or of microwaves. In the latter cases the intensity of sources decay exponentially with the distance from the surface of incidence.

The plan of the paper is the following: in Sect. 2 the problem is formulated and the basic equations are established. Then, in the main part of the paper, Sect. 3, similarity solutions are sought, focusing on several interesting cases, which lead to analytic solutions; in the final part of this section a "universal" solution is given. In Sect. 4, discussions and conclusions are presented.

2. Problem formulation and basic equations

We consider the steady Darcy free convection in a fluid saturated porous medium adjacent to a heated or cooled semi-infinite vertical flat plate of power-law temperature distribution (see Part I),

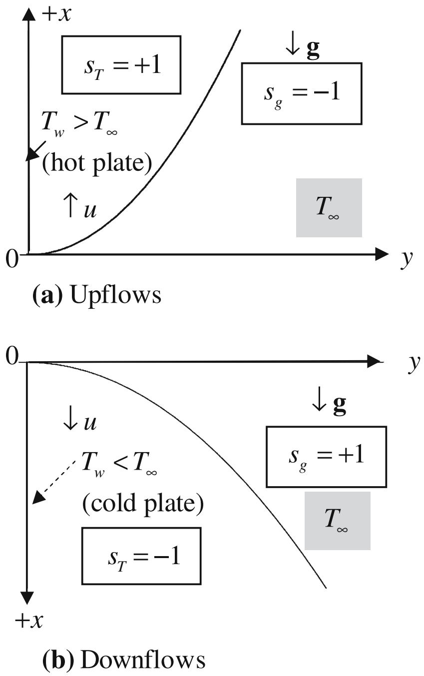
$$T_w(x) = T_\infty + s_T T_0 \left(\frac{x}{L}\right)^{\lambda}. \tag{1}$$

Here T_∞ denotes the ambient temperature of the saturated porous medium, $T_0 > 0$ specifies the temperature scale of the plate, L is a reference length and the sign function $s_T = \operatorname{sgn}(T_w - T_\infty)$ takes the value $s_T = +1$ for a "hot" plate, $T_w(x) > T_\infty$ and the value $s_T = -1$ for a "cold" one, $T_w(x) < T_\infty$. We further assume that in the porous medium continuously distributed heat sources of rate $S[W/m^3]$ are present.

Recalling several considerations made in Part I, the flow domain and the choice of the coordinate system are sketched in Figs. 1a, b where s_g denotes the projection of $\mathbf{g}/|\mathbf{g}|$ on the positive x-axis. Thus, $s_g = +1$ when the positive x-axis points in the direction of $\mathbf{g}$ (i.e., vertically downward) and $s_g = -1$ when it points in the direction opposite to $\mathbf{g}$ (i.e., vertically upward). In each of the two cases, depicted in Figs. 1a, b, the "forward", i.e., the usual boundary layer flows are considered, where the definite edge of the plate, $x = 0$, represents its leading edge. We notice that in both situations of Figs. 1a and b, the product of the sign functions s_T and s_g is the same

$$s_T s_g = -1. \tag{2}$$

Fig. 1a, b Representations of the free convection forward boundary layer up- and downflows over an upward projecting and downward projecting hot and cold plate, respectively. In the absence of the source term ($S = 0$) the two situations are physically equivalent. When, however $S \neq 0$, the up- and downflows in general become basically distinct



The basic equations of this problem are (see Part I)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3a}$$

$$u = -s_g \frac{g\beta K}{\upsilon} (T - T_\infty), \tag{3b}$$

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k_m \frac{\partial^2 T}{\partial y^2} + S, \tag{3c}$$

where one has assumed that the boundary-layer and the Boussinesq approximations hold. The plate is assumed impermeable, such that the boundary conditions of the problem associated with Eqs. (3a, b, c) are

$$\begin{aligned} v|_{y=0} &= 0, \qquad T|_{y=0} = T_w(x), \\ T|_{y\to\infty} &\to T_\infty, \quad S|_{y\to\infty} \to 0. \end{aligned} \tag{4a, b, c, d}$$

In Part I of the paper source terms of "type (I)" have been considered which depend only on the local temperature difference $T - T_\infty$, $S = S(T - T_\infty)$. The space distribution of this type of sources or sinks is not externally controlled and thus, not known *a priori*. The main goal of the present Part II of the paper is to consider the effect of the source terms of "type (II)", $S = S(x, y, z)$, which in contrast to the sources of type (I) are in general *externally controlled*, their space distribution and intensity being either *prescribed*, or they result from some physical laws which do not depend on temperature directly, as e.g., the heat release by absorption of infrared radiation or of microwaves.

3. Similarity solutions

3.1. Similarity reduction

With the usual definition of the stream function, $(u, v) = (\partial\psi/\partial y, -\partial\psi/\partial x)$ the basic Eq. (3) governing our free convection problem reduce to the single partial differential equation

$$\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial X\partial Y} - \frac{\partial\psi}{\partial X}\frac{\partial^2\psi}{\partial Y^2} = \alpha\left(\frac{\partial^3\psi}{\partial Y^3} + s_T\frac{RaL^2}{\rho c_p T_0}S(x, y)\right) \tag{5}$$

along with the boundary conditions

$$\left.\frac{\partial\psi}{\partial X}\right|_{Y=0} = 0, \qquad \left.\frac{\partial\psi}{\partial Y}\right|_{y=0} = \alpha RaX^{\lambda}, \qquad \left.\frac{\partial\psi}{\partial Y}\right|_{Y\to\infty} \to 0, \qquad S|_{Y\to\infty} \to 0 \tag{6a, b, c, d}$$

where $\alpha = k_m/(\rho c_p)$, Ra denotes the Darcy-Rayleigh number, $Ra = g\beta KT_0L/(\alpha\upsilon)$, and X and Y are the dimensionless coordinates $X = x/L$ and $Y = y/L$, respectively. The temperature field of the flow is given in terms of ψ by

$$T = T_\infty + \frac{s_T T_0}{\alpha Ra}\frac{\partial\psi}{\partial Y}. \tag{7}$$

The wall temperature distribution is obtained from Eqs. (7) and (6b) as

$$T_w(x) = T_\infty + s_T T_0 X^{\lambda}. \tag{8}$$

Thus, for $\lambda \neq 0$ the problem possesses a natural length scale L, namely L represents the distance x from the leading edge ($x = 0$) where the plate temperature has the prescribed value $T_w(L) = T_\infty + s_T T_0$.

A suitable general similarity transformation for the problem (5), (6) is [Eq. (45), Part I]

$$\psi(x, y) = \alpha\sqrt{Ra}X^{\frac{\lambda+1}{2}}f(\eta), \quad \eta = \sqrt{Ra}X^{\frac{\lambda-1}{2}}Y. \tag{9a, b}$$

Inserting Eqs. (9a, b) in Eqs. (5) and (6) furnishes the boundary value problem for the similar stream function f,

$$f''' + \frac{\lambda+1}{2}ff'' - \lambda f'^2 + s_T\frac{r_0}{Ra}X^{1-2\lambda}\tilde{S}(X, Y) = 0 \tag{10a}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{10b, c, d}$$

where the prime denotes differentiation with respect to η. In Eq. (10a) we have substituted

$$S(x, y) = S_0 \cdot \tilde{S}(X, Y) \quad \text{and} \quad r_0 = \frac{S_0L^2}{k_m T_0} \tag{11a, b}$$

where the constant $S_0\,[W/m^3]$ sets the intensity scale of the sources or sinks, $\tilde{S}(X, Y)$ is the dimensionless intensity, and r_0 is the (dimensionless) ratio of the flux density S_0L of the sources or sinks and the scale factor k_mT_0/L of the heat flux through the wall. Therefore, the contribution of the heat sources or sinks to the thermal energy balance of the flow is weighted by the ratio r_0/Ra of two characteristic quantities, r_0 and Ra.

Equation (10a) further shows that in the present case the necessary condition for existence of similar free convection flows is that the product $X^{1-2\lambda}\tilde{S}(X,Y)$ does not depend on the plate coordinate X, i.e., that the (dimensionless) source function $\tilde{S}(X,Y)$ is of the form

$$\tilde{S}(X,Y) = \frac{Ra}{r_0} G(\eta)\, X^{2\lambda-1}. \tag{12}$$

Here $G(\eta)$ is some (dimensionless) function of the similarity independent variable η and the factor Ra/r_0 has been included for further convenience. In this case Eq. (10a) of our basic boundary value problem reduces to the ordinary differential equation

$$f''' + \frac{\lambda+1}{2} ff'' - \lambda f'^2 + s_T G(\eta) = 0. \tag{13}$$

After the boundary value problem (13), (10b, c, d) has been solved (for a specified G), the similar velocity and temperature fields are obtained as

$$\begin{aligned} u &= \tfrac{\alpha}{L} Ra X^{\lambda} f'(\eta) \\ v &= -\tfrac{\alpha}{L}\sqrt{Ra} X^{\frac{\lambda-1}{2}}\left[\tfrac{\lambda+1}{2} f(\eta) + \tfrac{\lambda-1}{2}\eta f'(\eta)\right] \end{aligned} \tag{14a, b}$$

$$T = T_\infty + s_T T_0 X^{\lambda}\theta(\eta), \quad \theta(\eta) \equiv f'(\eta) \tag{15a, b}$$

and the dimensionless similar wall heat transfer coefficient results as

$$h = -s_T f''(0) \equiv -s_T \theta'(0) \tag{16}$$

Equations (14a) and (15) show that $f'(\eta)$ represents at the same time both the similar downstream velocity and the similar temperature profile $\theta(\eta) = f'(\eta)$.

We note that the source functions of type (12) can be expressed in terms of the wall temperature (8) by the relationship

$$\tilde{S}(X,Y) = \frac{Ra}{r_0}\left|\frac{T_w(x) - T_\infty}{T_0}\right|^{\frac{2\lambda-1}{\lambda}} G(\eta). \tag{17}$$

3.2. The case $\lambda = 1$ with $G(\eta) \sim e^{-a_1\eta}$

For the special choice of the similar source function

$$G(\eta) = s_T\left(1 - a_1^2\right)e^{-a_1\eta}, \quad a_1 > 0 \tag{18}$$

where a_1 is a positive dimensionless constant, the boundary value problem (13), (10b, c, d) admits for $\lambda = 1$ the exact analytical solution

$$f(\eta) = \frac{1}{a_1}\left(1 - e^{-a_1\eta}\right), \quad \theta(\eta) = e^{-a_1\eta}. \tag{19a, b}$$

It can be shown that (19) coincides with the solution (I-54) corresponding to $S = Q_0(T - T_\infty)$ and $\lambda = 1$, for

$$a_1 = \sqrt{1 - \frac{Q_0 L^2}{k_m Ra}}. \tag{20}$$

3.3. The case $\lambda = -1/3$

The special case $\lambda = -1/3$ of the power law exponent possesses the attractive property that the corresponding Eq. (13) can easily be integrated twice yielding the first order differential equation

$$f' = 1 + f''(0)\,\eta - s_T J(\eta) - \frac{1}{6} f^2 \tag{21}$$

where

$$J(\eta) = \int_0^{\eta} \left(\int_0^{\eta''} G(\eta')\, \mathrm{d}\eta' \right) \mathrm{d}\eta''. \tag{22}$$

Furthermore, by the change of dependent variable

$$f(\eta) = 6 \frac{\mathrm{d}}{\mathrm{d}\eta} [\ln W(\eta)] \tag{23}$$

the *nonlinear* Riccati type Eq. (21) reduces to the second order *linear* differential equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\eta^2} - \frac{1}{6}\left[1 + f''(0)\,\eta - s_T J(\eta)\right] W = 0. \tag{24}$$

Obviously, the two integration constants involved in the general solution of Eq. (24) must be chosen so that the similar stream function $f(\eta)$ obtained from Eq. (23) satisfies the boundary condition (10b) and its derivative obtained from Eq. (21) via Eqs. (24) and (23) satisfies the asymptotic condition (10d).

3.4. The case $\lambda = -1/3$ with $G(\eta) \sim e^{-a\eta}$

For a similar source function of the form

$$G(\eta) = s_G G_0\, e^{-a\eta}, \quad G_0, \quad a > 0, \quad s_G = \pm 1 \tag{25}$$

the integrals in Eq. (22) can easily be evaluated yielding

$$J(\eta) = \frac{s_G G_0}{a} \left(\eta + \frac{e^{-a\eta}}{a} - \frac{1}{a} \right). \tag{26}$$

Thus Eqs. (21) and (24) become

$$f' = \left(f''(0) - s_T s_G \frac{G_0}{a} \right) \eta + 1 + s_T s_G \frac{G_0}{a^2} \left(1 - e^{-a\eta}\right) - \frac{1}{6} f^2 \tag{27}$$

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\eta^2} - \frac{1}{6} \left[\left(f''(0) - s_T s_G \frac{G_0}{a} \right) \eta + 1 + s_T s_G \frac{G_0}{a^2} \left(1 - e^{-a\eta}\right) \right] W = 0. \tag{28}$$

A simple inspection of Eq. (27) shows that, as a direct consequence of the boundary condition (10d), the coefficient of the η-term must become zero, i.e.,

$$f''(0) = s_T s_G \frac{G_0}{a} \tag{29}$$

except for the case when the asymptotic behavior of $f(\eta)$ is such that

$$f \to \sqrt{6\left(f''(0) - s_T s_G \frac{G_0}{a}\right)\eta + 6\left(1 + s_T s_G \frac{G_0}{a^2}\right)} \quad \text{as } \eta \to \infty. \tag{30}$$

In the latter case, velocity and temperature solutions $f'(\eta) = \theta(\eta)$ may exist for which Eq. (29) does not hold. Obviously, these solutions must decay for algebraically as $\eta^{-1/2}$ when $\eta \to \infty$.

We first assume that Eq. (29) is satisfied. In this case Eq. (27) and the boundary condition (10d) imply for the similar entrainment velocity

$$f(\infty) = \sqrt{6\left(1 + s_T s_G \frac{G_0}{a^2}\right)}. \tag{31}$$

Accordingly, real solutions only exist for

$$1 + s_T s_G \frac{G_0}{a^2} \geq 0. \tag{32}$$

Furthermore, by the change of independent variable

$$z = \sqrt{s_T s_G \frac{2G_0}{3a^4}}\, e^{-\frac{a}{2}\eta} \tag{33}$$

Eq. (28) with condition (29) reduces to the equation of Bessel functions

$$z^2 \frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + z \frac{\mathrm{d}W}{\mathrm{d}z} + \left(z^2 - \nu^2\right) W = 0 \tag{34}$$

where

$$\nu = \sqrt{\frac{2}{3a^2}\left(1 + s_T s_G \frac{G_0}{a^2}\right)} = \frac{f(\infty)}{3a}. \tag{35}$$

The solution which satisfies the boundary condition (10b) for $s_T s_G = +1$ is

$$f(\eta) \equiv f_+(\eta) = 3az \frac{J'_\nu(z_0)\, J'_{-\nu}(z) - J'_{-\nu}(z_0)\, J'_\nu(z)}{J'_{-\nu}(z_0)\, J_\nu(z) - J'_\nu(z_0)\, J_{-\nu}(z)}, \quad (s_T s_G = +1) \tag{36}$$

where $J_{\pm\nu}(z)$ are the Bessel functions of the first kind, the primes denote differentiation with respect to z and $z_0 = z|_{\eta=0}$.

When $s_T s_G = -1$, the following two cases (compatible with condition (32)) must be distinguished.

(1) If $G_0 < a^2$ the solution is given by

$$f(\eta) \equiv f_{-,1}(\eta) = 3a\tilde{z} \frac{I'_\nu(\tilde{z}_0)\, I'_{-\nu}(\tilde{z}) - I'_{-\nu}(\tilde{z}_0)\, I'_\nu(\tilde{z})}{I'_{-\nu}(\tilde{z}_0)\, I_\nu(\tilde{z}) - I'_\nu(\tilde{z}_0)\, I_{-\nu}(\tilde{z})}, \quad (s_T s_G = -1, G_0 < a^2) \tag{37}$$

(2) If $G_0 = a^2$ the solution reduces to

$$f(\eta) \equiv f_{-,2}(\eta) = 3a\tilde{z} \frac{I_1(\tilde{z}_0)\, K_1(\tilde{z}) - K_1(\tilde{z}_0)\, I_1(\tilde{z})}{I_1(\tilde{z}_0)\, K_0(\tilde{z}) + K_1(\tilde{z}_0)\, I_0(\tilde{z})}, \quad (s_T s_G = -1, G_0 = a^2) \tag{38}$$

In Eqs. (37) and (38), $I_j\ (z)$ and $K_j\ (z)$ are the modified Bessel functions, see e.g., Abramowitz and Stegun (1964), and

$$\tilde{z} = \sqrt{\frac{2G_0}{3a^4}}\, e^{-\frac{a}{2}\eta}, \qquad \tilde{z}_0 = \tilde{z}|_{\eta=0} = \sqrt{\frac{2G_0}{3a^4}}. \tag{39a, b}$$

The corresponding similar velocities $f'\ (\eta)$ and temperature profiles $\theta\ (\eta)$are obtained from Eq. (27) as

$$f' = 1 + s_T s_G \frac{G_0}{a^2}\left(1 - e^{-a\eta}\right) - \frac{1}{6} f^2. \tag{40}$$

The similar wall heat flux $\theta'\ (0) = f''\ (0)$ is given by Eq. (29).

As an illustration, in Figs. 2 and 3 the similar stream functions $f_+\ (\eta)$ and $f_{-,2}\ (\eta)$ and the similar temperatures $\theta_+\ (\eta) = f'_+\ (\eta)$ and $\theta_{-,2}\ (\eta) = f'_{-,2}\ (\eta)$ corresponding to $s_T s_G = \pm 1$ and $G_0 = a = 1$ are plotted as functions of η. It is worth emphasizing that in the case $s_T s_G = -1$, the similar temperature $\theta_{-,2}\ (\eta)$ becomes negative for $\eta > 2.336$ (see Fig. 3).

We now turn to the case in which Eq. (29) is not satisfied, i.e., when in Eq. (27) the η-term is non-vanishing. In this case no exact solutions are available. Numerical investigations show, however, that solutions do exist and, as it should be, they decay according to Eq. (30) algebraically as $\eta^{-1/2}$ when $\eta \to \infty$. For $G_0 = a = 1$ e.g., we find solutions for any value $f''\ (0) > +1$ when $s_T s_G = +1$, and any $f''\ (0) < -1$ when $s_T s_G = -1$. As $f''\ (0) \to +1$ and $f''\ (0) \to -1$ these families of slowly decaying solutions go over in the respective solutions $f'_+\ (\eta) = \theta_+\ (\eta)$ and $f'_{-,2}\ (\eta) = \theta_{-,2}\ (\eta)$ plotted in Fig. 3. These results are illustrated in Figs. 4 and 5.

3.5. A universal solution

The solution of our free convection boundary value problem (13), (10b, c, d) depends in general on the value of the power law exponent λ in an essential way. The actual form of

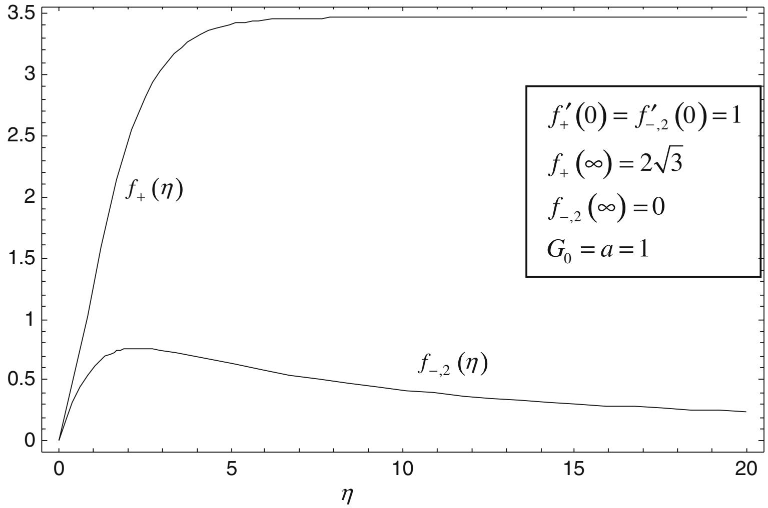


Fig. 2 The similar stream functions $f_+\ (\eta)$ and $f_{-,2}\ (\eta)$ corresponding to $s_T s_G = \pm 1$ and $G_0 = a = 1$, plotted as functions of η. The curve $f_{-,2}\ (\eta)$ reaches its maximum $\left(f_{-,2}\right)_{\max} = 0.762$ at $\eta = 2.336$

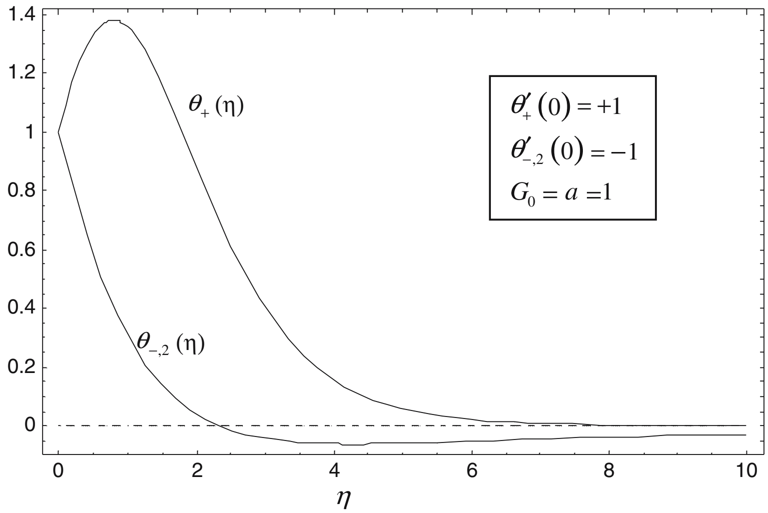


Fig. 3 The similar temperatures $\theta_+(\eta) = f'_+(\eta)$ and $\theta_{-,2}(\eta) = f'_{-,2}(\eta)$ corresponding to $s_T s_G = \pm 1$ and $G_0 = a = 1$, plotted as functions of η. The curve $\theta_{-,2}(\eta)$ crosses the η-axis at $\eta = 2.336$ and reaches its negative (!) minimum $\left(\theta_{-,2}\right)_{\min} = -0.06173$ at $\eta = 4.279$. The curve $\theta_+(\eta)$ reaches at $\eta = 0.787$ its maximum $(\theta_+)_{\max} = 1.382$

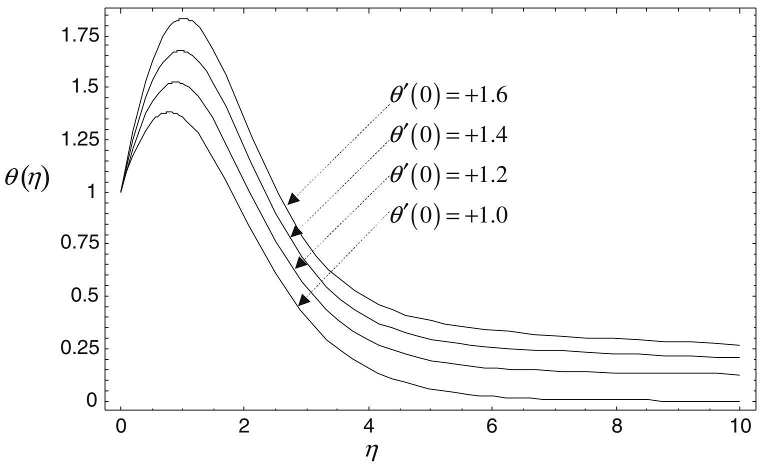


Fig. 4 The similar temperatures $\theta(\eta) = f'(\eta)$ corresponding to $s_T s_G = +1$, $G_0 = a = 1$ and different values of $\theta'(0) = f''(0)$ plotted as functions of η. One sees that as $f''(0) \to +1$, the family of the slowly decaying solutions goes over in the solution $\theta_+(\eta) = f'_+(\eta)$ which satisfies Eq. (29)

the solution is always the result of a subtle interplay between the internal heat generation or absorption, heat transport by convection and diffusion and the heat transfer between the wall and the convecting fluid. In this sense it is quite surprising that even a λ-dependent source term can exert such an influence on the above mentioned heat exchange processes that the resulting similarity solution becomes independent of the main parameter λ of the problem. But this can actually be the case. Indeed, choosing the similar source function as

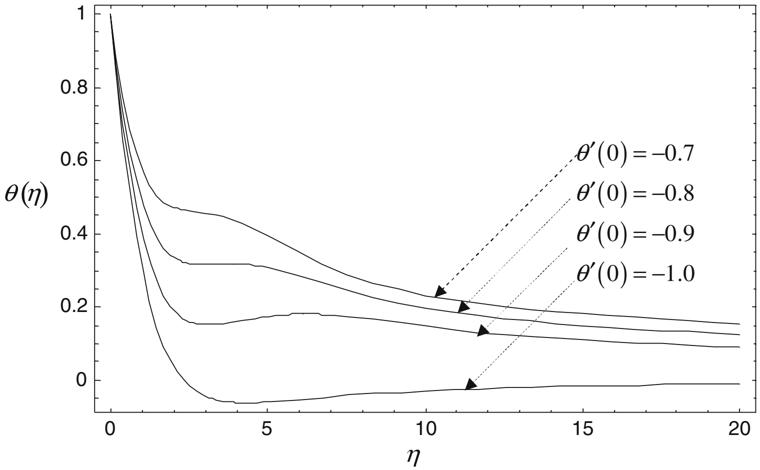


Fig. 5 The similar temperatures $\theta(\eta) = f'(\eta)$ corresponding to $s_T s_G = -1$, $G_0 = a = 1$ and different values of $\theta'(0) = f''(0)$ plotted as functions of η. One sees that as $f''(0) \to -1$, the family of the slowly decaying solutions goes over in the solution $\theta_{-,2}(\eta) = f'_{-,2}(\eta)$ which satisfies Eq. (29)

$$G(\eta) = s_T \frac{\lambda b^4 + (\lambda + 1) b^3 \eta - 6b^2}{(b + \eta)^4} \tag{41}$$

where b is a positive constant, the problem (13), (10b, c, d) admits the "universal solution"

$$f(\eta) = \frac{b\eta}{b + \eta}, \quad \theta(\eta) = \frac{b^2}{(b + \eta)^2} \tag{42a, b}$$

which is independent of λ. The corresponding wall heat transfer coefficient (16) depends only on the value of b and is given by

$$h = -s_T \theta'(0) = \frac{2 s_T}{b}. \tag{43}$$

The particular case $\lambda = -1$ is especially interesting since the corresponding Eq. (13),

$$f''' + f'^2 + s_T G(\eta) = 0 \tag{44}$$

does not admit solutions which satisfy the boundary conditions (10b, c, d) if the source term $G(\eta)$ is absent. If however $G(\eta)$ has the corresponding form (41), i.e.,

$$G(\eta) = -s_T \frac{b^4 + 6b^2}{(b + \eta)^4} \quad (\lambda = -1) \tag{45}$$

one obtains the solution (42).

4. Summary and Conclusions

In the above Sections the effect of some source terms of "type (II)", $S = S(x, y, z)$, on the self-similar free convection flows over a vertical plate has been examined. As shown in Sect. 3.1, the (power law) similarity reduction of the basic partial differential equations is possible when the source term is of the general form (12), i.e.,

$$S(x, y, z) \sim G(\eta)\, X^{2\lambda - 1}, \quad \eta = \sqrt{Ra}\, X^{\frac{\lambda - 1}{2}}\, Y \tag{46a, b}$$

where $G(\eta)$ is an arbitrary function of the similarity independent variable η, being only subject to the asymptotic condition

$$G(\eta) \to 0 \quad \text{as } \eta \to \infty \tag{47}$$

which, in agreement with Eq. (6d), ensures that $S \to 0$ as $\eta \to \infty$.

A physically important point is that the corresponding boundary value problem (13), (10b, c, d) depends on the sign s_T of the plate temperature. This circumstance implies that the source term breaks the usual upflow or downflow symmetry of the free convection problem, i.e., the flows over an upward projecting hot plate and its downward projecting cold counterpart become physically distinct.

In order to be specific, the case of sources or sinks described by G functions of exponential type

$$G(\eta) = s_G G_0\, e^{-a\,\eta}, \quad G_0, \quad a > 0, \quad s_G = \pm 1 \tag{48}$$

has been examined in some detail. As mentioned also in Part I, this kind of similar source function G has already been used by Crepeau and Clarksean (1997) in connection with the viscous flow of incompressible clear fluids, as well as by Postelnicu and Pop (1999), Postelnicu et al. (2000), Grosan and Pop (2001, 2002) and Bagai (2003) in connection with the free convection over a vertical flat plate or over a body of arbitrary shape embedded in a fluid-saturated porous medium by using the Darcy or non-Darcy flow models. The main contribution of the present paper to the investigation of sources or sinks of type (48) is the presentation of exact analytical solutions for the cases $\lambda = 1$ (Sect. 3.2) and $\lambda = -1/3$ (Sect. 3.3 and 3.4). It is worth emphasizing again that in the case $\lambda = 1$ the effect of the source term of type (II) specified by Eqs. (46) and (48) is equivalent to that of a source term of type (I) which depends linearly on the local temperature difference, $S = Q_0\,(T - T_\infty)$ (see Sect. 3.2). In the case $\lambda = -1/3$, the solution of the problem shows a sensible dependence on the sign combination $s_T s_G$, as discussed in Sect. 3.4 and illustrated in Figs. 2–5.

Concerning the similar source functions of the "popular" exponential form (48), some comments are in order. First of all, we underline that the exponential form (48) is not a necessary requirement of the similarity (which holds for arbitrary G's). This mathematically "friendly" source function is also suitable to describe the heat generation phenomenon which accompanies the absorption in the fluid saturated porous medium of infrared or microwave radiation emitted by the plate. As it is well known, the decrease of the radiation intensity due to absorption in the volume of a homogeneous material is governed by the exponential law

$$I(y) = I_0 \cdot e^{-\alpha_{abs}\, y} \tag{49}$$

where α_{abs} is the absorption coefficient and y is the depth inside the material measured from the surface of incidence. The energy of the absorbed infrared radiation is then added to the kinetic energy of the atoms or molecules of the absorbent material. This means that Eq. (49) describes at the same time also the rate of volumetric heat generation in the medium. Now, in the case $\lambda = 1$ the source term (46), (48) has the form

$$S \sim X \cdot e^{-a\sqrt{Ra}\,Y}. \tag{50}$$

Comparing Eq. (50) to (49) we see that a source term of the form (50) is adequate to describe the volumetric heat generation by absorption of the infrared radiation in the case in which the surface intensity of emission increases linearly with the plate coordinate x, while the absorption coefficient is constant. For an arbitrary λ we obtain instead of (50) a relationship of the form

$$S = S_0 (X) \cdot e^{-\alpha_{abs}(X)Y} \tag{51}$$

with the surface intensity distribution $S_0 (X) \sim X^{2\lambda-1}$ and a coordinate dependent absorption coefficient $\alpha (X) \sim X^{(\lambda-1)/2}$.

In the last section of the paper, Sect. 3.5 the similar source function (41) has been considered which, in contrast to the previous exponential cases, represents a rational function of the similarity independent variable η, and which depends on the exponent λ, too. We did not succeed in finding a practical application of this type of source. It has rather a basic significance, showing that for any given λ (i.e., a given plate temperature distribution) there always exists a source distribution (41) in such a way that the similar flow and temperature fields specified by Eqs. (42a, b) are independent of λ, i.e., universal. This strange effect is the reason for including Sect. 3.5 in our paper.

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